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On the applicability of the frozen flux approximation in core flow modeling as a function of temporal frequency and spatial degree

Stefan Maus[1]

Stefan Maus, National Geophysical Data Center, NOAA E/GC1, 325 Broadway, Boulder, CO 80305-3328, USA, email: Stefan.Maus@noaa.gov

[1]CIRES at the University of Colorado and NOAA's National Geophysical Data Center

Abstract. The secular variation of the geomagnetic field is due to flow of molten iron in the Earth's core. One can invert for the flow at the core-surface, assuming that the magnetic field is frozen into the core fluid. The validity of the frozen flux approximation (FFA) depends on the scale length of the magnetic field and the period of core motions. A first-order quantitative assessment can be made by solving the induction equation for a spherical conductor oscillating in an ambient magnetic field. The period at which the FFA holds strongly decreases with increasing spherical harmonic degree of the ambient field. For degrees smaller than five, the flux of the ambient magnetic field can be considered as frozen-in for periods of more than a thousand years. For ambient magnetic fields of spherical harmonic degree 10 and higher, the FFA still holds up to periods of about a hundred years. However, the real situation in the core could be more complex if the field cannot be assumed as ambient, but is generated by the flow itself.

1. Introduction

With a new generation of high-accuracy satellite magnetic missions in low-Earth orbit, there is a revived interest in inferring core-surface flow from observed time variations of the geomagnetic field [*Hulot et al.*, 2002; *Voorhies*, 2004; *Eymin and Hulot*, 2005; *Holme and Olsen*, 2006]. An intriguing application of core flow modeling is that it may improve the forecast of the secular variation by advecting the present geomagnetic field forward in time (Maus, Silva and Hulot, manuscript submitted, preprint available at http://info.geomag.us/Smaus/Doc/flow.pdf).

Central to core-surface flow inversions is the assumption that magnetic flux is carried along with the core fluid, as if it were frozen in. This is equivalent to neglecting magnetic diffusion. With further simplifying assumptions, one can then invert for a surface flow explaining the observed changes in the radial component of the geomagnetic field at the core-mantle boundary. The frozen flux approximation (FFA) was first used by *Roberts and Scott* [1965] for core-surface flow inversions. Using scaling relations, they argued that the length scale of the core field (assumed 1000 km) divided by the velocity of core flow (assumed 1 mm/s) gives a period of 30 years, which is small compared with the free decay time-scale of core fields (order of 10,000 years).

Bloxham and Jackson [1991] reviewed further scaling arguments and discussed possibilities of testing the FFA on real data. Studying periodic magnetic field variations for a horizontal core-surface flow above a convective volume flow, *Braginsky and Le Mouel* [1993] found that an inversion adopting the FFA provides an estimate of the vertical average of the horizontal velocity in the top layer. *Gubbins* [1996] proposes a formalism to include diffusion in core-surface flow mapping from geomagnetic observations, cautioning however, that this exacerbates the non-uniqueness of the inverse problem. *Gubbins and Kelly* [1996] and *Love* [1999] address the importance of the time scale of the flow, pointing out that a steady flow driving a steady geodynamo could not be determined by adopting the FFA. The validity of the FFA has also been investigated in numerical dynamo simulations by *Roberts and Glatzmaier* [2000], *Rau et al.* [2000] and *Amit et al.* [2007]. While the remoteness of the parameter regimes from the real Earth is of some concern, the results imply that the FFA is useful to some extent in recovering core surface flow. In summary, these studies seem to indicate that the FFA is justified when relating core flow and geomagnetic field variations at short enough time scales and large enough spatial scales. The present study is aimed at clarifying these dependencies on a simple model.

The geomagnetic field is a superposition of contributions of different scale length, as can be described by a spherical harmonic expansion. By comparing a spherical harmonic coefficient with its secular variation, one obtains an estimate of the period of the corresponding field variations [*Hulot and Le Mouël*, 1994]. Characteristic periods are of the order of thousands of years for the lowest degrees, decreasing to decades for degrees 10 and higher. From the perspective of geomagnetic field modeling, it would be valuable to know up to what periods the FFA holds for a given spherical harmonic degree of the field. Here, I propose a simple first-order quantitative assessment, by solving the induction equation for a spherical conductor oscillating in an ambient magnetic field of spherical harmonic degree ℓ and order m. As expected, the percentage of advected field is found to increase with conductivity, decrease with the oscillation period, and decrease with the spherical harmonic degree of the field.

The solutions of the induction equation in a spherical conductor are of course well-known and have been given, for example, by*Lamb* [1883], *Chapman and Bartels* [1940] and *Bullard* [1949]. In particular, the equations given by *Parkinson and Hutton* [1989, Chapter 2.3] could be adopted to the present purpose by introducing the appropriate formal analogies. Instead, a simple derivation of the solution in Backus' notation [*Backus et al.*, 1996] is provided here.

2. Solution of the induction equation

In the following, the induction equation shall be solved for a simple Earth-like situation to obtain a first-order quantitative assessment of the expected error made in adopting the FFA. The model is that of a uniform core performing whole-body rotational oscillations in an arbitrary ambient magnetic field. The induction equation for a uniform spherical conductor separates into toroidal and poloidal parts. The toroidal field is confined to the conductor, so we are only interested in the poloidal part.

2.1. Differential Equation

Following the notation of *Backus et al.* [1996] the poloidal magnetic field can be written as

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p(\mathbf{r}, t), \tag{1}$$

where $\boldsymbol{\Lambda}$ is the surface curl and $p(\mathbf{r}, t)$ is a poloidal field scalar which depends on location $\mathbf{r}$ and time t. Then the induction equation for the poloidal field within the sphere [*Backus et al.*, 1996, 5.4.13] is

$$\partial_t p(\mathbf{r}, t) = \eta \nabla^2 p(\mathbf{r}, t), \tag{2}$$

where ∂_t denotes the temporal derivative and $\eta = 1/\mu_0\sigma$, with permeability of vacuum μ_0 and constant conductivity σ. Expanding $p(\mathbf{r},t)$ into spherical harmonics [*Backus et al.*, 1996, p. 197] gives

$$p(\mathbf{r},t) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi), \tag{3}$$

with spherical harmonic coefficients $p_\ell^m(r,t)$ and spherical harmonic basis functions $\beta_\ell^m(\vartheta,\varphi)$, using polar coordinates with radius r, colatitude ϑ and longitude φ. The following derivations are valid for any desired normalization of the spherical harmonic basis. Inserting the spherical harmonic representation (3) into the induction equation (2) and further using

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2} \tag{4}$$

yields a differential equation for the spherical harmonic coefficients of the poloidal field scalar as

$$\partial_t p_\ell^m(r,t) = \eta[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}]p_\ell^m(r,t). \tag{5}$$

2.2. Model

Consider a homogeneous, conducting sphere in an ambient magnetic field which is constant in time. If the sphere is at rest, the magnetic field will (eventually) permeate the entire sphere. Now let us rotate the sphere back and forth with some frequency ω. It is advisable to solve the induction problem for a reference frame co-moving with the material body. If the amplitude of the rotational oscillations is small compared with the scale length of the spatial magnetic field variations, the variation of the magnetic field for a co-moving observer is linearly related to the longitudinal variation. Since the basis functions $\beta_\ell^m(\vartheta,\varphi)$ have a $e^{im\varphi}$ longitudinal dependence, we then have

$$\delta[p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi)] = imp_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi)\,\delta\varphi. \tag{6}$$

For the longitudinal variation due to rotational oscillation we can now assume

$$\delta\varphi = \delta_0 e^{-i\omega t}, \tag{7}$$

where δ_0 is the amplitude of the rotational oscillation in degrees or radians. The apparent time variations of the ambient field, as seen by an observer on the sphere, are therefore given by

$${}_e p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi) = im\delta_0 e^{-i\omega t}{}_0 p_\ell^m \left(\frac{r}{a}\right)^\ell \beta_\ell^m(\vartheta,\varphi), \tag{8}$$

where a is the radius of the sphere, ${}_e p_\ell^m(r,t)$ denotes the poloidal scalar of the external field seen in the co-moving frame and ${}_0 p_\ell^m$ are the corresponding constant coefficients of the ambient field in the inertial frame.

2.3. Boundary conditions

The time varying external field, as seen in the co-moving frame, induces toroidal currents in the conducting sphere. In the co-moving frame these currents can be interpreted as preventing the external field variations from penetrating into the sphere. The success of this response depends on the conductivity of the sphere. For a perfect conductor, the induced magnetic field completely cancels the inducing field inside of the sphere.

In the inertial frame, the secondary field can be interpreted as the advected part of the ambient field. For a perfect conductor, the entire field is advected back and forth in the oscillation, and the field remains constant inside the sphere. The interesting question of course is: What happens for an intermediate conductor?

Returning to the co-moving frame, the total magnetic field external to the sphere can be regarded as the superposition of the ambient source field ${}_e p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi)$ and the secondary induced field ${}_i p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi)$ with sources inside the sphere. Note that both are time-varying and do not have to be in-phase. Outside of the sphere, the ambient and induced fields fulfill Laplace's equation and therefore have radial behavior

$$ {}_e p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi) = {}_e p_\ell^m(t)\left(\frac{r}{a}\right)^\ell \beta_\ell^m(\vartheta,\varphi) \tag{9} $$

$$ {}_i p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi) = {}_i p_\ell^m(t)\left(\frac{a}{r}\right)^{\ell+1} \beta_\ell^m(\vartheta,\varphi), \tag{10} $$

where a is the radius of the sphere. In the following section, we seek a solution $p_\ell^m(r,t)\beta_\ell^m(\vartheta,\varphi)$ inside of the sphere. On the boundary, this solution has to match the sum of the ambient and induced fields. Furthermore, the radial derivative has to be continuous at the surface of the sphere [*Backus et al.*, 1996, eq. 5.3.26]. This gives two boundary conditions

$$ p_\ell^m(a,t) = {}_e p_\ell^m(t) + {}_i p_\ell^m(t) \tag{11} $$

$$ r\partial_r p_\ell^m(r,t)|_{r=a} = \ell\, {}_e p_\ell^m(t) - (\ell+1)\, {}_i p_\ell^m(t) \tag{12} $$

2.4. Solution inside of the sphere

The poloidal field scalar inside of the sphere has to fulfill the differential equation (5) which can be solved using the ansatz

$$ p_\ell^m(r,t) = Ae^{-i\omega t}(\kappa r)^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(\kappa r), \tag{13} $$

where A and κ are complex constants and $J_{\ell+\frac{1}{2}}$ are Bessel functions of the first kind [*Gradshteyn and Ryzhik*, 1994]. Then the radial derivative is

$$ r\partial_r p_\ell^m(r,t) = Ae^{-i\omega t} r\partial_r (\kappa r)^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(\kappa r) \tag{14} $$

$$= Ae^{-i\omega t}[-\frac{1}{2}(\kappa r)^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(\kappa r) + (\kappa r)^{\frac{1}{2}}\partial_{\kappa r}\, J_{\ell+\frac{1}{2}}(\kappa r)]. \tag{15}$$

Using *Gradshteyn and Ryzhik* [1994, 8.472.1], the radial derivative of a Bessel function can be expressed as

$$\partial_{\kappa r}\, J_{\ell+\frac{1}{2}}(\kappa r) = J_{\ell-\frac{1}{2}}(\kappa r) - \frac{\ell+\frac{1}{2}}{\kappa r}J_{\ell+\frac{1}{2}}(\kappa r). \tag{16}$$

Inserting (16) into (15) gives

$$r\partial_r p_\ell^m(r,t) = Ae^{-i\omega t}[(\kappa r)^{\frac{1}{2}}J_{\ell-\frac{1}{2}}(\kappa r) - (\ell+1)(\kappa r)^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(\kappa r)]. \tag{17}$$

Using (13) and (17) for the left sides in the boundary conditions (11) and (12) then gives two equations

$$Ae^{-i\omega t}(\kappa a)^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(\kappa a) \;=\; {}_ep_\ell^m(t) + {}_ip_\ell^m(t) \tag{18}$$

$$Ae^{-i\omega t}[(\kappa a)^{\frac{1}{2}}J_{\ell-\frac{1}{2}}(\kappa a) - (\ell+1)(\kappa a)^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}(\kappa a)] \;=\; \ell\,{}_ep_\ell^m(t) - (\ell+1)\,{}_ip_\ell^m(t) \tag{19}$$

which can be combined to eliminate $Ae^{-i\omega t}$ as

$$\kappa a J_{\ell-\frac{1}{2}}(\kappa a)\,{}_ip_\ell^m(t) = [-\kappa a J_{\ell-\frac{1}{2}}(\kappa a) + (2\ell+1)J_{\ell+\frac{1}{2}}(\kappa a)]\,{}_ep_\ell^m(t). \tag{20}$$

Using the relation [*Gradshteyn and Ryzhik*, 1994, 8.471.1]

$$(2\ell+1)J_{\ell+\frac{1}{2}}(\kappa a) = [\kappa a J_{\ell-\frac{1}{2}}(\kappa a) + \kappa a J_{\ell+\frac{3}{2}}(\kappa a)], \tag{21}$$

we then obtain the complex transfer function

$$\frac{{}_ip_\ell^m(t)}{{}_ep_\ell^m(t)} = \frac{J_{\ell+\frac{3}{2}}(\kappa a)}{J_{\ell-\frac{1}{2}}(\kappa a)}. \tag{22}$$

This function has a modulus and a phase. Here, only the modulus, relating the amplitudes of the ambient and induced fields, is of interest. Note that the amplitude A does not appear in this equation, meaning that the transfer function is independent of m, δ_0, and ${}_0p_\ell^m$, as long as $m \neq 0$ and the oscillation amplitude δ_0 is small compared with the longitudinal wavelength of the ambient field.

Now we still need to find the value of κ, which is obtained by inserting the Ansatz (13) into the differential equation (5), giving

$$[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2} + \frac{i\omega}{\eta}](\kappa r)^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(\kappa r) = 0, \tag{23}$$

which can be rearranged to

$$[\partial_{\kappa r}^2 + \frac{2}{\kappa r}\partial_{\kappa r} + (\frac{i\omega/\eta}{\kappa^2} - \frac{\ell(\ell+1)}{(\kappa r)^2})](\kappa r)^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(\kappa r) = 0. \tag{24}$$

Considering that the functions $z^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(z)$ fulfill the spherical Bessel equation [*Abramowitz and Stegun*, 1972, 10.1.1]

$$[\partial_z^2 + \frac{2}{z}\partial_z + (1 - \frac{\ell(\ell+1)}{z^2})] z^{-\frac{1}{2}} J_{\ell+\frac{1}{2}}(z) = 0, \tag{25}$$

it follows that

$$\kappa = \pm\sqrt{i\omega/\eta} = \pm\sqrt{i\omega\mu_0\sigma} \tag{26}$$

$$= \pm e^{\pi/4}\sqrt{\omega\mu_0\sigma}, \tag{27}$$

where the transfer function (22) takes on identical values for the positive and the negative values of κ. The parameter κ corresponds to the parameter k in the related solution of *Gubbins* [1996, eq. 15]. In the classical electromagnetic induction problem [*Parkinson and Hutton*, 1989, Chapter 2], the parameter κ is known as the propagation constant.

3. Results and discussion

From the transfer function (22) we can now infer the ratio of the induced to the inducing field. If the ratio is low, the core motions fail to transport the magnetic flux. For a ratio close to unity, on the other hand, most of the field is advected and the FFA holds. For our simple model, the percentage of advected field depends on only four parameters: conductivity σ, core radius a, motion period $2\pi/\omega$ and spherical harmonic degree n of the ambient field. The percentage of advected flow does not depend on the strength of the ambient field. It also does not depend on the amplitude of the rotational oscillation and the order m of the ambient magnetic field, provided that $m \neq 0$ and the amplitude of the rotational oscillation is small compared with the longitudinal wavelength of the ambient field. The latter condition is necessary because a larger amplitude of the motions would mean that the conductor passes through several peaks and troughs of the ambient field, which corresponds to a higher oscillation frequency. This situation leads to the mathematical complications pointed out by *Bullard* [1949, p. 434].

The percentage of advected field as a function of core motion period is plotted in Figure 1 for different spherical harmonic degrees of the ambient field. In this, a core conductivity of $3 \times 10^5 S/m$ is assumed. It is seen that the FFA holds well for a low-degree magnetic field for core motions with periods up to about thousand years. For ambient magnetic fields of degree 10 and higher, the field can still be assumed as frozen-in for periods up to around one hundred years. At these higher degrees, however, an additional complication arises in core flow inversions, due to the masking of the main field by the crustal field at degrees higher than 13. Interaction between the flow and the unknown higher-degree core field contributes significantly to the observed secular variation at smaller degrees [*Hulot et al.*, 1992]. The resulting ambiguity caused by main field truncation has also been confirmed in experiments to recover core flow from geodynamo simulations [*Roberts and Glatzmaier*, 2000; *Rau et al.*, 2000; *Amit et al.*, 2007]. In practice, errors due to main field truncation could dominate over uncertainties caused by the adoption of the FFA.

To the first approximation given by our simple model, the FFA appears justified in core flow inversions, considering the characteristic time periods of the observed secular variation. This interpretation is challenged by *Love* [1999] who argues that the geomagnetic field is generated by a predominantly steady dynamo, where advection of the field by steady flow is balanced by diffusion. This steady flow does not generate time variations of the magnetic field and is therefore invisible in the usual frozen flux inversions. He further argues that cross-coupling between the steady and time-varying velocity and field terms would make it impossible to even infer the shorter periods of the flow. In terms of our simple model, which assumes an imposed ambient field, this scenario corresponds to the presence of core motions with long periods, for which the FFA does not hold (right side in Figure 1). Since the contributions at different periods are additive, it would still be possible to invert for shorter period variations of the flow, even in the presence of an invisible steady flow. The situation may however be different when the magnetic field cannot be considered as ambient, but is actually generated by the flow.

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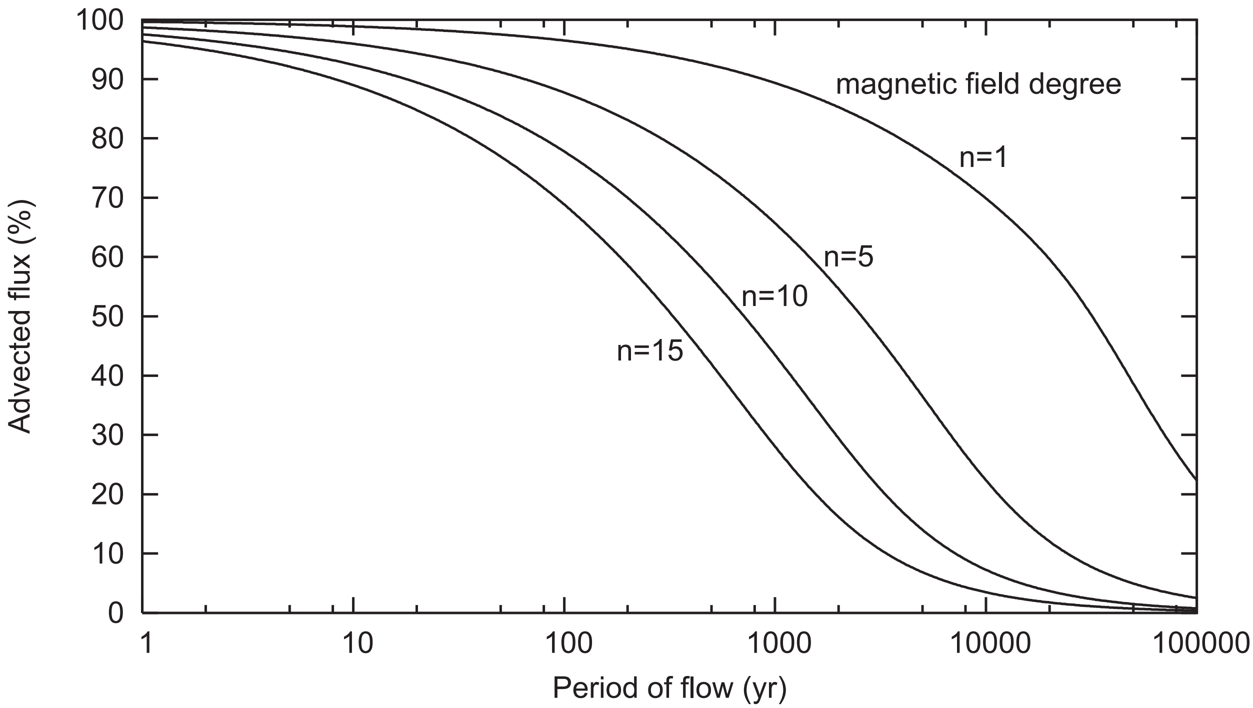

Figure 1. Percentage of advected field as a function of core-motion period for different degrees of the ambient field, assuming a core conductivity of $3 \times 10^5 S/m$.